

POSITIONAL GRAPHS AND CONDITIONAL STRUCTURE OF WEAKLY NULL SEQUENCES

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ABSTRACT. We prove that, unless assuming additional set theoretical axioms, there are no reflexive space without unconditional sequences of density the continuum. We give for every integer n there are normalized weakly-null sequences of length ω_n without unconditional subsequences. This together with a result of [Do-Lo-To] shows that ω_ω is the minimal cardinal κ that could possibly have the property that every weakly null κ -sequence has an infinite unconditional basic subsequence. We also prove that for every cardinal number κ which is smaller than the first ω -Erdős cardinal there is a normalized weakly-null sequence without subsymmetric subsequences. Finally, we prove that mixed Tsirelson spaces of uncountable densities must always contain isomorphic copies of either c_0 or ℓ_p , with $p \geq 1$.

1. INTRODUCTION

Maurey and Rosenthal [Ma-Ro] were the first to construct a weakly null sequence $(x_n)_{n < \omega}$ without infinite unconditional subsequences. Their machinery of coding block-sequences of finite subsets of ω by integers and using this to define special functionals has become a standard tool in this area for constructing examples of separable Banach spaces with conditional norms. The most famous of these is the example of Gowers and Maurey [Ga-Ma] of a separable reflexive space without unconditional basic sequences. In [Ar-Lo-To] a new coding of finite block-sequences of finite subsets of ω_1 was employed in building a reflexive space of density ω_1 with no infinite unconditional basic subsequence. While at that time it was not clear if the construction can be stretched up to ω_2, ω_3 , etc, the paper [Do-Lo-To] showed that there could be no such construction for density ω_ω , at least if one is not willing to go beyond the standard axioms of set theory. More precisely, [Do-Lo-To] shows that it is consistent that every weakly null sequence of length ω_ω must contain an infinite unconditional basic subsequence. In the present paper we supplement this by extending the Maurey-Rosenthal construction and showing that for every integer n there are normalized weakly-null sequences of length ω_n without infinite unconditional basic subsequences. This is achieved by a deeper analysis of the combinatorial properties of families of finite subsets of ω_n that could be used in coding the conditionality inside a norm on $c_{00}(\omega_n)$. We express this using the notion of countable chromaticity for certain positional graphs on $[\omega_n]^{<\omega}$, a notion that seems of independent interest and that might have other uses.

Recall that Argyros and Talias [Ar-To] have constructed another example of a non separable Banach space without infinite unconditional basic sequences. Their space has density continuum

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and is a dual of a separable hereditarily indecomposable space and so highly non-reflexive. We explain this by showing that it is consistent that every weakly null sequence of length continuum (and so, in particular, every reflexive space of density continuum) must contain an infinite unconditional basic subsequence. This is done by connecting the problem with other classical combinatorial ideas such as, for example, free subsets of algebraic structures, or the possibility of extending the Lebesgue measure to all sets of reals, known as the Banach problem.

Recall also that subsymmetric sequences $(x_n)_n$ (see Section 2) are those that for any $m_0 < \dots < m_k$ and $n_0 < \dots < n_k$ the linear extension of $x_{m_i} \mapsto x_{n_i}$ defines an isomorphism of norm uniformly bounded. It is a classical result that every non-trivial weakly-null subsymmetric sequence has an unconditional subsequence. So, modulo an application of Rosenthal's ℓ_1 -theorem, getting a subsymmetric basic sequence is just a step away from getting an unconditional one. While analyzing the corresponding cardinal conditions on densities that guarantee the existence of each such sequence, we have discovered the huge difference between these two notions. For example, we show that while a relatively simple argument of Ketonen [Ke] shows that every space of density at least an ω -Erdős cardinal must contain an infinite sub-symmetric sequence, the converse is also true: For every cardinal κ which is not ω -Erdős cardinal there is a weakly null κ -sequence without infinite subsymmetric subsequences.

As it is well known in this area, the Tsirelson space [Ts] was the first example of a space without subsymmetric sequences. In the final section of our paper we explain this construction when lifted into the realm of non separable spaces. For example, we show that so-called mixed Tsirelson spaces of uncountable densities, unlike their separable counterparts, must always contain subsymmetric sequences, indeed isomorphic copies of either c_0 or ℓ_p , for some $p \geq 1$.

2. PRELIMINARIES AND NOTATION

Given a set X , and a cardinal number λ , let $[X]^\lambda$ denote the collection of all subsets of X of cardinality λ . Let $[X]^{\leq \lambda}$ and $[X]^{< \lambda}$ denote the collection of subsets of X of cardinality $\leq \lambda$ and $< \lambda$, respectively.

Let κ be a cardinal number. Given $s, t \in [\kappa]^{< \omega}$, we write $s < t$ to denote that $\max s < \min t$. Given an integer n , let $\text{Bs}_n(\kappa)$ denote the collection of all *block sequences* of length n of finite subsets of κ , i.e. all sequences $(s_i)_{i < n}$ of finite subsets $s_i \subseteq \kappa$ such that $s_i < s_{i+1}$ for every $i < n - 1$. Let $\text{Bs}_{< \omega}(\kappa)$ denote the set of all finite block sequences of finite subsets of κ .

A family \mathcal{B} of subsets of κ is *large* when $\mathcal{B} \cap [A]^n \neq \emptyset$ for every infinite subset A of κ and every $n < \omega$. The family \mathcal{B} is *very-large* when for every infinite subset A of κ there is some infinite subset B of A such that $[B]^{< \omega} \subseteq \mathcal{B}$.

Recall that the order-type $\text{otp}(W, R)$ of a well ordered set (W, R) is the unique ordinal number α such that there is an order-preserving bijection between (W, R) and α when α is endowed with its natural order. Given $A, B \subseteq \kappa$ of the same order-type, let $\vartheta_{A,B} : A \rightarrow B$ be the unique order-preserving bijection between A and B . Given $I \subseteq \text{otp}(A)$, let $s[I] := \vartheta_{\text{otp}(A), A}'' I$, and given $i \in \text{otp}(A)$, let $(A)_i := A[\{i\}]$.

Recall that a seminormalized sequence $(x_\alpha)_{\alpha < \gamma}$ in a Banach space $(X, \|\cdot\|)$ indexed in an ordinal number γ is called a (Schauder) basic sequence when there is a constant $K \geq 1$ such

that

$$\left\| \sum_{\alpha < \lambda} a_\alpha x_\alpha \right\| \leq K \left\| \sum_{\alpha < \gamma} a_\alpha x_\alpha \right\|$$

for every $\lambda < \gamma$ and every sequence of scalars $(a_\alpha)_{\alpha < \gamma}$.

A basic sequence $(x_\alpha)_{\alpha < \gamma}$ is called (suppression) unconditional when there is a constant $K \geq 1$ such that

$$\left\| \sum_{\alpha \in A} a_\alpha x_\alpha \right\| \leq K \left\| \sum_{\alpha < \gamma} a_\alpha x_\alpha \right\|$$

for every $A \subseteq \gamma$ and every sequence of scalars $(a_\alpha)_{\alpha < \gamma}$.

The sequence $(x_\alpha)_{\alpha < \gamma}$ is called subsymmetric when there is a constant $K \geq 1$ such that for every $s, t \subseteq \gamma$ of the same cardinality and every sequence $(a_n)_{n < |s|}$ of scalars have that

$$\frac{1}{K} \left\| \sum_{n < |s|} a_n x_{\vartheta_{|s|,s}(n)} \right\| \leq \left\| \sum_{n < |s|} a_n x_{\vartheta_{|t|,t}(n)} \right\| \leq K \left\| \sum_{n < |s|} a_n x_{\vartheta_{|s|,s}(n)} \right\|.$$

3. CHROMATIC NUMBERS OF POSITIONAL GRAPHS

In this section we define a positional graph whose vertex set is a sufficiently large family of finite subset of some cardinal κ and show that a good coloring of this graph with countably many colors can be used to define a normed space with a weakly null κ -sequence with no infinite unconditional subsequence.

Definition 3.1. Let $\mathfrak{G} = (V, E)$ be a graph. The chromatic number $\chi(\mathfrak{G})$ is the minimal cardinal number κ such that there is a coloring $c : V \rightarrow \kappa$, called a *good coloring of \mathfrak{G}* , of the set of vertexes V into κ -many colors such that no two vertexes $v_0 \neq v_1$ in an edge in E have the same color c .

EXAMPLE 3.2. Let $\mathfrak{G}_{\text{card}}(\kappa) := (V, E_{\text{card}})$ be the graph with vertexes V the set of all finite block sequences $(s_i)_{i < d}$, and let E be the set of pairs $((s_i)_{i < d}, (t_i)_{i < l})$ such that

$$(|s_i|)_{i < d} \neq (|t_i|)_{i < l}.$$

Then $\mathfrak{G}_{\text{card}}(\kappa)$ has countable chromatic number: Let $d : \omega^{<\omega} \rightarrow \omega$ be any 1-1 function, and for a finite block sequence $(s_i)_{i < d}$ in κ define $c((s_i)_{i < d}) := d((|s_i|)_{i < d})$. Clearly c is a good coloring.

Recall that for $A, B \subseteq \kappa$ we write $A \sqsubseteq B$ when A is initial part of B , i.e. when $A \subseteq B$ and $A < B \setminus A$.

Definition 3.3. Given an integer n and two subsets A and B of κ , we say that A and B are in n - Δ -position when there are I, J subsets of $A \cap B$ such that

- (a) $I < J$, $A \cap B = I \cup J$ and $|J| \leq n$.
- (b) $I \sqsubseteq A, B$.

A and B are in Δ -position when they are in 0- Δ -position. Let $\mathfrak{G}_n(\kappa)$ be the graph whose set of vertexes is $[\kappa]^{<\omega}$, and the set of edges E_{pos} is the family of pairs (s, t) such that s and t are not in n - Δ -position. Given a family $\mathcal{B} \subseteq [\kappa]^{<\omega}$, we write $\mathfrak{G}_n(\mathcal{B})$ to denote the restriction of the graph $\mathfrak{G}_n(\kappa)$ to set of vertexes \mathcal{B} .

It is easily seen (see, for example, [To], Chapter 3) that $\chi(\mathfrak{G}_0(\omega)) = \chi(\mathfrak{G}_0(\omega_1)) = \omega$ but that $\chi(\mathfrak{G}_0(\omega_2)) > \omega$. In fact, we have the following more general result

Theorem 3.4. $\chi(\mathfrak{G}_{n-2}(\omega_n)) > \omega$ for all $2 \leq n < \omega$

PROOF. This follows from the standard fact (see, for example, [Di-To]) that for every $c : [\omega_n]^n \rightarrow \omega$ there exist a sequence $s_0 < s_1 < \dots < s_{n-1}$ of 2-element subsets of ω_n such that c is constant on their product. \square

The use of the polarized partition property in this proof suggest the following fact proved along the same lines.

Theorem 3.5. If $\kappa \rightarrow \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \lambda \end{pmatrix}^{<\omega}$ then $\chi(\mathfrak{G}_n(\kappa)) > \lambda$ for all $n < \omega$.

This partition property that has been analyzed in detail in [Di-To] asserts that for every coloring $c : [\kappa]^{<\omega} \rightarrow \lambda$ there is an infinite block sequence $(s_i)_{i < \omega}$ of 2-element subsets of κ such that c is constant on $\prod_{i < n} s_i$ for all $n < \omega$. Cardinals κ with this partition property with $\lambda = 2$ are called *polarized cardinals*. It is known that (see, for example, [Di-To]) that every polarized cardinal satisfies the partition property for $\lambda = 2^{\aleph_0}$, so in particular, every polarized cardinal is greater than continuum. It is also known (see [Di-To]), that a polarized cardinal may not be greater than 2^{\aleph_1} and that, more interestingly, that it is consistent that \aleph_ω is a polarized cardinal. Clearly if \aleph_ω is a polarized cardinal then it is the minimal polarized cardinals. The minimal polarized cardinals are interesting because they satisfy the partition property for $\lambda = \theta^{\aleph_0}$ for every $\theta < \kappa$. So, in particular, the minimal polarized cardinal κ has the property that $\theta^{\aleph_0} < \kappa$ for all $\theta < \kappa$ (see [Di-To]). Their interest for us here comes from the following immediate fact.

Corollary 3.6. If κ is the minimal polarized cardinal then $\chi(\mathfrak{G}_n(\kappa)) = \kappa$ for all $n < \omega$.

Corollary 3.7. It is consistent relative to the existence of a measurable cardinal that for every $n < \omega$ one has that $\chi(\mathfrak{G}_n(\omega_\omega)) = \omega_\omega$.

The following result shows that ω_ω is in some sense the minimal cardinal that could possibly have this property.

Theorem 3.8. For every $n < \omega$ there is a very-large $\mathcal{B}_n \subseteq [\omega_n]^{<\omega}$ such that $\chi(\mathfrak{G}_{2n-1}(\mathcal{B}_n)) = \omega$.

We postpone its proof to the Section 5.

4. POSITIONAL GRAPHS AND CONDITIONAL NORMS

The purpose of this section is to prove the following result that connects positional graphs with the existence of large weakly null sequences with no infinite unconditional basic subsequences.

Theorem 4.1. Suppose that for some cardinal κ and some $n < \omega$ there is a very-large family $\mathcal{B} \subseteq [\kappa]^{<\omega}$ such that $\chi(\mathfrak{G}_n(\mathcal{B})) = \omega$. Then there is a norm $\|\cdot\|$ on $c_{00}(\kappa)$ such that the sequence $(u_\gamma)_{\gamma < \kappa}$ of unit vectors of $c_{00}(\kappa)$ is a weakly null Schauder basis of the completion of $(c_{00}(\kappa), \|\cdot\|)$ with no infinite unconditional basic subsequences.

PROOF. As with basically all known constructions of conditional norms, the basic trick of adding conditionality remains unchanged since its first appearance in the classical construction by Maurey and Rosenthal [Ma-Ro] of a weakly-null ω -sequence without infinite unconditional subsequences. Let $n < \omega$, let \mathcal{B} be a very-large family on κ such that $\mathfrak{G}_n(\mathcal{B}) = \omega$ and let c be a good coloring of the graph $(\mathcal{B}, E_{\text{card}} \cup E_{\text{pos}})$. By re-enumeration if needed, we assume that c takes values on a set $M \subseteq \omega \setminus n$ with the lacunary condition

$$\sum_{m \in M} \sum_{l \in M \setminus \{m\}} \min\left\{\sqrt{\frac{l}{m}}, \sqrt{\frac{m}{l}}\right\} \leq 1, \quad (1)$$

and such that $c(\emptyset) \geq n$. We say that a finite block sequence $(s_i)_{i < d}$ of subsets of κ is \mathcal{B} -special when

- (a) $\bigcup_{j < i} s_j \in \mathcal{B}$ for every $i < d$.
- (b) $|s_i| = c(\bigcup_{j < i} s_j)$ for every $i < d$.

Claim 4.1.1. *For every infinite set A of κ and every d there is a \mathcal{B} -special sequence of length d consisting on subsets of A .*

Proof of Claim: Let B be an infinite subset of A such that $[B]^{<\omega} \subseteq \mathcal{B}$. Now we simply choose any sequence $(s_i)_{i < d}$ consisting of subsets of B such that (b) above holds. \square

Now let

$$K := \left\{ \sum_{i < d} \frac{1}{\sqrt{|s_i|}} \mathbb{1}_{s_i} : (s_i)_{i < d} \text{ is } \mathcal{B}\text{-special} \right\}.$$

On $c_{00}(\kappa)$ define the norm $\|\cdot\|_K$ for $x \in c_{00}(\kappa)$ by

$$\|x\|_K := \max\{\|x\|_\infty, \sup\{\langle x, f \rangle : f \in K\}\}.$$

Let X be the completion of $(c_{00}(\kappa), \|\cdot\|_K)$. It is not difficult to prove that $(u_\gamma)_{\gamma < \kappa}$ is a Schauder basis of X .

Claim 4.1.2. *Suppose that $s \in [\kappa]^{<\omega}$ is such that $|s| \in M$. Then*

$$\left\| \frac{1}{|s|^{\frac{1}{2}}} \mathbb{1}_s \right\| \leq 2. \quad (2)$$

Consequently, $(u_\gamma)_{\gamma < \kappa}$ is a weakly-null sequence.

Proof of Claim: Let $(s_i)_{i < d}$ be \mathcal{B} -special with $d \leq |s_0|$. Then

$$\left\langle \frac{1}{|s|^{\frac{1}{2}}} \mathbb{1}_s, \sum_{i < d} \frac{1}{|s_i|^{\frac{1}{2}}} \right\rangle \leq \sum_{i < d, |s_i| < |s|} \frac{|s_i|^{\frac{1}{2}}}{|s|^{\frac{1}{2}}} + \sum_{i < d, |s_i| > |s|} \frac{|s|^{\frac{1}{2}}}{|s_i|^{\frac{1}{2}}} + 1 \leq 2.$$

Now if $(u_\gamma)_\gamma$ were not weakly null, we could find $\varepsilon > 0$ and an infinite set $A \subseteq \omega_n$ such that

$$\left\| \frac{1}{|s|^{\frac{1}{2}}} \mathbb{1}_s \right\| \geq \varepsilon |s|^{\frac{1}{2}} \text{ for every } s \subseteq A \text{ finite.} \quad (3)$$

Clearly (3) is in contradiction with (2). \square

Claim 4.1.3. *$(u_\gamma)_{\gamma < \kappa}$ does not have infinite unconditional subsequences.*

Proof of Claim: Fix a subset $A \subseteq \kappa$ of order type ω , and fix $L \geq 1$. We see that $(u_\gamma)_{\gamma \in A}$ is not L -unconditional. Let $k < \omega$ be such that $k > 8L$. Let $x := \sum_{i < k} (-1)^i \mathbb{1}_{s_i} / |s_i|^{1/2}$, where $(s_i)_{i < k}$ is a special sequence in A , and let $y = \sum_{i < k/2} \mathbb{1}_{s_{2i}} / |s_{2i}|^{1/2}$. Since $f := \sum_{i < k} \mathbb{1}_{s_i} / |s_i|^{1/2} \in K$, it follows that $\|y\|_K \geq \langle f, y \rangle \geq k/2$. We are going to see that $\|x\|_K \leq 4$. To this end, fix $g = \sum_{i < d} \mathbb{1}_{t_i} / |t_i|^{1/2} \in K$. Let

$$m_0 := \max\{i < \min\{d, k\} : |s_i| = |t_i|\}.$$

It follows then that $c(s_0, \dots, s_{m_0-1}) = c(t_0, \dots, t_{m_0-1})$, and hence $|s_i| = |t_i|$ for all $i < m_0$ and, setting $s = \bigcup_{i < m_0} s_i$ and $t = \bigcup_{i < m_0} t_i$, there are $I, J \subseteq s \cap t$ such that

- (a) $I < J$, $s \cap t = I \cup J$ and $|J| < n$.
- (b) $I \sqsubseteq s, t$.

Let $i_0 < m_0$ be the last $i < m_0$ such that $s_i \subseteq I$. Then it follows that

- (c) $s_i = t_i$ for every $i \leq i_0$, and
- (d) $(\bigcup_{i=i_0+1}^{m_0-1} s_i) \cap (\bigcup_{i=i_0+1}^{m_0-1} t_i)$ has cardinality at most $n - 1$.

Since $|s_i| = c(s_0, \dots, s_{i-1})$, $i < k$ and $|t_j| = c(t_0, \dots, t_{j-1})$, $j < d$, it follows that $|s_i| \neq |t_j|$ for $i \neq j$. Hence,

$$\begin{aligned} |\langle g, x \rangle| &\leq \left| \sum_{i < d} \sum_{j < k, |t_j| = |s_i|} \left\langle \frac{1}{\sqrt{|t_i|}} \mathbb{1}_{t_i}, \frac{(-1)^j}{\sqrt{|s_j|}} \mathbb{1}_{s_j} \right\rangle \right| + \sum_{i < d} \sum_{j < k, |t_j| \neq |s_i|} \left| \left\langle \frac{1}{\sqrt{|t_i|}} \mathbb{1}_{t_i}, \frac{(-1)^j}{\sqrt{|s_j|}} \mathbb{1}_{s_j} \right\rangle \right| \leq \\ &\leq \left| \sum_{i \leq i_0} \left\langle \frac{1}{\sqrt{|s_i|}} \mathbb{1}_{s_i}, (-1)^i \frac{1}{\sqrt{|s_i|}} \mathbb{1}_{s_i} \right\rangle \right| + \left| \sum_{i=i_0+1}^{m_0-1} \left\langle \frac{1}{\sqrt{|t_i|}} \mathbb{1}_{t_i}, (-1)^i \frac{1}{\sqrt{|s_i|}} \mathbb{1}_{s_i} \right\rangle \right| + \\ &+ \left| \left\langle \frac{1}{\sqrt{|t_{m_0}|}} \mathbb{1}_{t_{m_0}}, (-1)^i \frac{1}{\sqrt{|s_{m_0}|}} \mathbb{1}_{s_{m_0}} \right\rangle \right| + \sum_{i < d} \sum_{j < k, |t_j| \neq |s_i|} \left| \left\langle \frac{1}{\sqrt{|t_i|}} \mathbb{1}_{t_i}, \frac{(-1)^j}{\sqrt{|s_j|}} \mathbb{1}_{s_j} \right\rangle \right| \leq \\ &\leq 1 + \frac{|\bigcup_{i=i_0+1}^{m_0-1} s_i \cap \bigcup_{i=i_0+1}^{m_0-1} t_i|}{|s_{i_0}|} + 2 \leq 4. \end{aligned}$$

□

□

REMARK 4.2. The previous example leads to a c_0 -saturated space. It is possible to modify the construction to get a reflexive example, tough unconditionally saturated.

Problem 1. *Is there a similar combinatorial condition on uncountable κ that ensures the existence of a reflexive Banach space of density κ with no infinite unconditional basic sequences?*

In [Ar-Lo-To], we have provided such an example for $\kappa = \omega_1$.

5. PROOF OF THEOREM 3.8

Recall that the *Shift graph* on a totally ordered set $(A, <)$ is the graph $([A]^{<\omega}, \text{Sf})$ whose edges are the pairs (s, t) of finite subsets of κ such that $s \setminus \{\min s\} \sqsubseteq t$. The following notions play an important role in the proof.

Definition 5.1. Let A, X be two totally ordered sets, and let $f : [A]^n \rightarrow X$.

(a) We call f *Shift-increasing* if

$$f(s) < f(t) \text{ for every } (s, t) \in \text{Sf in } A. \quad (4)$$

(b) We call f *min-dependant* if

$$f(s) = f(t) \text{ implies that } \min s = \min t \text{ for every } s, t \in [A]^n. \quad (5)$$

The proof crucially depends also on the following concept from [To].

Definition 5.2. A function $\varrho : [\kappa^+]^2 \rightarrow \kappa$ is called an (injective version of) ϱ -function if

(a) ϱ is subadditive, i.e. for every $\alpha < \beta < \gamma < \kappa^+$

$$(a.1) \quad \varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\},$$

$$(a.2) \quad \varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\alpha, \gamma)\}.$$

(b) $\varrho(\alpha, \beta) \neq \varrho(\bar{\alpha}, \beta)$ for every $\alpha \neq \bar{\alpha} < \beta$.

(c) $\varrho(\alpha, \beta) \neq \varrho(\beta, \gamma)$ for every $\alpha < \beta < \gamma$.

It is proved in [To] (see Definition 3.2.1, Lemma 3.2.2 dealing with the case $\kappa = \omega$ and Chapter 9 for the general version) that such a function $\varrho : [\kappa^+]^2 \rightarrow \kappa$ exists for every regular cardinal κ .

Definition 5.3. For each integer n we fix an injective ϱ function $\varrho^{(n)}$ on ω_n . Let $n \in \omega$. For each $i \leq n$ we define recursively $f_i^{(n)} : [\omega_n]^{i+1} \rightarrow \omega_{n-i}$ as follows:

$$(1) \quad f_0^{(n)} := \text{Id}_{\omega_n};$$

$$(2) \quad f_i(\alpha_0, \alpha_1, \dots, \alpha_i) := \varrho^{(n-(i-1))}(f_{i-1}(\alpha_0, \dots, \alpha_{i-1}), f_{i-1}(\alpha_1, \dots, \alpha_i)) \text{ for each } \alpha_0 < \dots < \alpha_i \text{ in } \omega_n \text{ and each } 0 < i \leq n.$$

Let $f_n := f_n^{(n)} : [\omega_n]^{n+1} \rightarrow \omega$.

Proposition 5.4. Suppose that $\alpha, \bar{\alpha} < \alpha_0 < \dots < \alpha_{i-1}$ are such that

$$(a) \quad f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = f_i^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-1}).$$

$$(b) \quad f_j^{(n)}(\alpha, \alpha_0, \dots, \alpha_{j-1}), f_j^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{j-1}) < f_j^{(n)}(\alpha_0, \dots, \alpha_j) \text{ for every } j < i.$$

Then $\alpha = \bar{\alpha}$.

PROOF. This is done by induction on $i \geq 0$. The case $i = 0$ is trivial. Suppose that $i > 0$. Then

$$\begin{aligned} \varrho^{(n-(i-1))}(f_{i-1}^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1})) &= f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = \\ &= f_i^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-1}) = \varrho^{(n-(i-1))}(f_{i-1}^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1})). \end{aligned}$$

By the hypothesis (b), it follows that

$$f_{i-1}^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-2}) < f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}).$$

So, by the property (b) of $\varrho^{(n-(i-1))}$ in Definition 5.2, we get that $f_{i-1}^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-2}) = f_{i-1}^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-2})$, and by inductive hypothesis we obtain that $\alpha = \bar{\alpha}$. \square

Lemma 5.5. For every $A \subseteq [\omega_n]^\omega$ there is $B \in [A]^\omega$ such that $f_i^{(n)} \upharpoonright [B]^{i+1}$ is Shift-increasing for every $i \leq n$.

PROOF. Let $c : [A]^{n+2} \rightarrow {}^{n+1}\mathbf{3}$ be the coloring defined for each $\alpha_0 < \dots < \alpha_{n+1}$ in A by

$$(c(\alpha_0, \dots, \alpha_{n+1}))_i = \begin{cases} 0 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) < f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 1 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) = f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 2 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) > f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \end{cases} \quad (6)$$

for each $i \leq n$. Using the Ramsey Theorem we can find $B \in [A]^\omega$ which is c -monochromatic, with constant value $(\varepsilon_i)_{i \leq n} \in {}^{n+1}\mathbf{3}$.

Claim 5.5.1. $\varepsilon_i = 0$ for every $i \leq n$.

It follows easily from the claim that B fulfills the requirement of the statement in the Lemma. So, it rests to show the claim.

Proof of Claim: We prove first that $\varepsilon_i < 2$ for every $i \leq n$. Otherwise, fix $i \leq n$ such that $\varepsilon_i = 2$, and let $(s_n)_{n \in \omega}$ be an arbitrary sequence in $[B]^{n+2}$ such that for every n one has that $(s_n, s_{n+1}) \in \text{Sf}_{n+2}$. For each n , let $t_n := s_n[i+1]$. It follows then that $f_i^{(n)}(t_n) > f_i^{(n)}(t_{n+1})$ for every n . This is impossible because the range of $f_i^{(n)}$ is the well founded set ω_{n-i} .

Next, we prove that $\varepsilon_i \neq 1$ for every $i \leq n$, by induction on $i \leq n$. For $i = 0$ we have that $f_0^{(n)}(\alpha) = \alpha$ so it is clear that $\varepsilon_0 = 0$. Now suppose that $i > 0$. Let $\alpha_0 < \dots < \alpha_{i+1}$ in B . By inductive hypothesis,

$$f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}) < f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i) < f_{i-1}^{(n)}(\alpha_2, \dots, \alpha_{i+1}). \quad (7)$$

It follows then by the property (c) of $\varrho^{(n-(i-1))}$ that

$$\begin{aligned} f_i^{(n)}(\alpha_0, \dots, \alpha_i) &= \varrho^{(n-(i-1))}(f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}), f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i)) \neq \\ &\neq \varrho^{(n-(i-1))}(f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i), f_{i-1}^{(n)}(\alpha_2, \dots, \alpha_{i+1})) = f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}), \end{aligned}$$

so $\varepsilon_i \neq 1$. □

□

Lemma 5.6. For every $A \subseteq [\omega_n]^\omega$ there is $B \in [A]^\omega$ such that the restriction $f_i^{(n)} \upharpoonright [B]^{i+1}$ is min-dependant for every $i \leq n$.

PROOF. We use first Lemma 5.5 to find $C \in [A]^\omega$ such that $f_i^{(n)} \upharpoonright [C]^{i+1}$ are Shift-increasing for every $i \leq n$. We use now the Erdős-Rado Canonization Theorem to find an infinite set $B \in [C]^\omega$ and for each $i \leq n$ sets $J_i \subseteq i+1$ such that

$$f_i^{(n)}(s) = f_i^{(n)}(t) \text{ if and only if } s[J_i] = t[J_i] \text{ for every } s, t \in [B]^{i+1} \text{ and every } i \leq n. \quad (8)$$

Claim 5.6.1. $0 \in J_i$ for every $i \leq n$.

It is clear that this claim proves that C has the desired properties.

Proof of Claim: Fix $i \leq n$. Since (8) holds, it suffices to prove that if $f_i^{(n)}(\alpha_0, \alpha_1, \dots, \alpha_i) = f_i^{(n)}(\bar{\alpha}_0, \alpha_1, \dots, \alpha_i)$, then $\alpha_0 = \bar{\alpha}_0$. Since $f_j^{(n)} \upharpoonright [B]^{j+1}$ are all Shift-increasing, we can apply Proposition 5.4 and get that $\alpha_0 = \bar{\alpha}_0$, so $0 \in J_i$. □

□

Definition 5.7. Let \mathcal{B}_n be the set of all finite sets s of ω_n such that

- (a) $f_n \upharpoonright [s]^{n+1}$ is min-preserving.
- (b) $f_i^{(n)} \upharpoonright [s]^{i+1}$ is shift-increasing for every $i < n$.

Let $c_n : \mathcal{B}_n \rightarrow \bigcup_{k < \omega} {}^{k^{n+1}}\omega$ be the coloring

$$c_n(s) := f_n \circ \vartheta_{[s],s}^{n+1}.$$

Proposition 5.8. \mathcal{B}_n is a very-large family of finite subsets of ω_n and c_n is a good coloring of the graph $\mathfrak{G}_n(\mathcal{B}_n)$.

PROOF. The Lemma 5.5 and Lemma 5.6 proves that \mathcal{B} is very-large. Let us see that c_n is a good coloring. Suppose that $s, t \in \mathcal{B}$ are such that $c_n(s) = c_n(t)$. If $|s \cap t| < 2n$, then s and t are in n - Δ -position. So, suppose that $|s \cap t| \geq 2n$. Let $J := \{\gamma_0 < \dots < \gamma_{n-1} < \gamma_n < \dots < \gamma_{2n-1}\}$ be the set of the last $2n$ elements of $s \cap t$. We prove first that $\vartheta(\gamma_i) = \gamma_i$ for every $i < n$: Fix such $i < n$; then

$$f_n(\gamma_i, \gamma_n, \dots, \gamma_{2n-1}) = f_n(\vartheta_{s,t}(\gamma_i), \vartheta_{s,t}(\gamma_n), \dots, \vartheta_{s,t}(\gamma_{2n-1})) \quad (9)$$

Since $\vartheta_{s,t}''(s \cap t) \cup t$, and since $f_n \upharpoonright [t]^{n+1}$ is min-dependant, it follows from (9) that $\vartheta_{s,t}(\gamma_i) = \gamma_i$.

Claim 5.8.1. $s \cap \gamma_0 = t \cap \gamma_0 = s \cap t \cap \gamma_0$.

It is clear that the previous claim gives that s and t are in n - Δ -position.

Proof of Claim: Let $\gamma \in s \cap \gamma_0$. Then

$$f_n(\gamma, \gamma_0, \dots, \gamma_{n-1}) = f_n(\vartheta_{s,t}(\gamma), \vartheta_{s,t}(\gamma_0), \dots, \vartheta_{s,t}(\gamma_{n-1})) = f_n(\vartheta_{s,t}(\gamma), \gamma_0, \dots, \gamma_{n-1}). \quad (10)$$

Since $f_i^{(n)} \upharpoonright [s]^{i+1}$ and $f_i^{(n)} \upharpoonright [t]^{i+1}$ are shift-increasing for every $i < n$, it follows from Proposition 5.4 that $\vartheta_{s,t}(\gamma) = \gamma$, so $\gamma \in t$. Similarly one proves that $t \cap \gamma_0 \subseteq s \cap \gamma_0$. \square

\square

6. WEAKLY-NULL SEQUENCES AND THE CONTINUUM

Recall that at the present there exist only two examples of non separable spaces with no infinite unconditional basic sequences, the example of [Ar-To] that has of density 2^{\aleph_0} and the example of [Ar-Lo-To] that has density \aleph_1 . While the space of [Ar-To] has density that it is at least consistently larger there is a crucial differences between these two examples. The space of [Ar-Lo-To] is reflexive while the space of [Ar-To] is far from this, as it is a dual of a separable hereditarily indecomposable space. In this section we use some known combinatorial properties of cardinals to show that this difference is indeed essential. More precisely, we show that there could be no constructions of weakly null sequences of length continuum with no unconditional subsequences if one is not willing to use additional set theoretic assumptions such as, for example, the Continuum Hypothesis. This follows from the following result.

Theorem 6.1. *It is consistent relative to the consistency of the existence of an ω -Erdős cardinal¹ that every weakly null sequence of length continuum contains an infinite unconditional basic sequence.*

¹See the next section for definition.

Our proof will also reveals the following interesting connection with the classical problem of Banach about extending the Lebesgue measure to all sets of reals.

Theorem 6.2. *Suppose that the Lebesgue measure extends to a total countably additive measure on \mathbb{R} . Then every weakly null sequence of length continuum contains an infinite unconditional basic sequence.*

Our analysis is based on the following classical concept.

Definition 6.3. A cardinal κ has the ω -free-set property if every algebra \mathcal{A} on domain A of cardinality κ with no more than countably many operations (but with no restriction on their arities) has an infinite free set, an infinite subset X of A such that no $x \in X$ is in the subalgebra generated by $X \setminus \{x\}$. We use $\text{Fr}(\kappa, \omega)$ to denote this property of a cardinal κ .

It is known that every polarized cardinal κ has the free-set property $\text{Fr}(\kappa, \omega)$ (see [Di-To]) and that it is this property that is more closely tied with the problem of finding infinite unconditional subsequences of a given weakly null κ -sequence. More precisely, we have the following result from [Do-Lo-To].

Theorem 6.4. *If $\text{Fr}(\kappa, \omega)$ holds then every weakly null κ -sequence contains an infinite unconditional basic subsequence.*

Unlike the first polarized cardinal, the first cardinal satisfying $\text{Fr}(\kappa, \omega)$ does not need to be larger than the continuum. There are several ways one can see this, the simplest being the following fact which must have been already observed before but we were not able to find a reference.

Lemma 6.5. *The property $\text{Fr}(\kappa, \omega)$ is preserved under forcing by the posets satisfying the countable chain condition.*

PROOF. Given a poset \mathcal{P} and a \mathcal{P} -name $\dot{\mathcal{A}}$ for an algebra on κ , we may assume that the operations of $\dot{\mathcal{A}}$ are all coded up in a single function of the form $\dot{f} : \kappa^{<\omega} \rightarrow \kappa$. Since \mathcal{P} satisfies the countable chain condition, for every name τ for an ordinal $< \kappa$ there is a countable set $C(\tau) \subseteq \kappa$ such that every $p \in \mathcal{P}$ forces τ is an element of $C(\tau)$. So we can find a sequence $g_n : \kappa^{<\omega} \rightarrow \kappa$ ($n < \omega$) with the property that for every $s \in \kappa^{<\omega}$

$$C(\dot{f}(s)) = \{g_n(s) : n < \omega\}.$$

Applying our assumption $\text{Fr}(\kappa, \omega)$ to the algebra $(\kappa, g_n)_{n < \omega}$ we get an infinite subset X of κ that is free in this algebra. It follows that every $p \in \mathcal{P}$ forces that X is also free relative to the algebra $\dot{\mathcal{A}}$ in the forcing extension of \mathcal{P} . \square

Proof of Theorem 6.1. Every ω -Erdős cardinal has the property $\text{Fr}(\kappa, \omega)$ (see, for example, [De]). Now the conclusion follows from Theorem 6.4 and Lemma 6.5, by going to the forcing extension where at least κ many real numbers are added by a poset satisfying the countable chain condition. \square

Proof of Theorem 6.2. This follows by combining Theorem 6.4 with the classical results of Erdős and Hajnal [Er-Ha] and Solovay [So]. More precisely, by the result of [So], the assumption implies the existence of a regular cardinal $\kappa \leq 2^{\aleph_0}$ with the *Jonsson property*, i.e., with the property that every algebra with domain κ contains a proper subalgebra of the same cardinality. On the other hand, it is proved in [Er-Ha] (see also [De]) that every Jonsson cardinal κ has the free-set property $\text{Fr}(\kappa, \omega)$. \square

One of the reasons for mentioning Theorem 6.2 is that it suggests using measure theoretic conditions on norms in order to have infinite unconditional subsequences of a given large weakly null sequence. This, of course, remains to be explored, but we mention now one of such possible results.

Proposition 6.6. *Suppose that $(x_\alpha)_{\alpha < \kappa}$ be a normalized weakly null sequence² in some normed space X . Suppose that the index-set κ supports a countably additive probability measure μ that gives measure zero to all countable subsets of κ and is defined on some σ -field of subsets of κ . Suppose that all norm-configurations induced by subsets of the finite power κ^n are measurable relative to the power measure μ^n . Then $(x_\alpha)_{\alpha < \kappa}$ contains an infinite unconditional basic subsequence.*

PROOF. For each finite set $s \subseteq \kappa$, choose a countable subset $N_s \subseteq S_{X^*}$ such that $\|x\| = \sup\{f(x) : f \in N_s\}$ for every $x \in \langle x_\alpha \rangle_{\alpha \in s}$. Let $\theta : \text{FIN}(\kappa) \rightarrow [\kappa]^\omega$ be defined for $s \in \text{FIN}(\kappa)$ by

$$\theta(s) := \{\alpha < \kappa : \text{there is some } f \in N_s \text{ such that } f(x_\alpha) \neq 0\}.$$

A finite set $s \in \text{FIN}(\kappa)$ is called θ -free if

$$f(t) \cap s \subseteq t \text{ for every } t \subseteq s.$$

Let $F_n \subseteq [\kappa]^n$ be the set of θ -free sequences of cardinality n . We identify a finite set $s \in \text{FIN}(\kappa)$ with the strictly increasing enumeration sequence $\vartheta_{|s|,s} \in \kappa^n$.

Claim 6.6.1. *Let μ be a σ -additive measure on κ , and let μ_n denote the product measure of μ on κ^n . If the set F_n is μ_n -measurable, then $\mu_n([\kappa]^n \setminus F_n) = 0$.*

Proof of Claim: Suppose otherwise that $\mu_n([\kappa]^n \setminus F_n) > 0$. For each $I \subsetneq n$, and $j \in n \setminus I$, let

$$S_{I,j} := \{s \in [\kappa]^n : (s)_j \in f(s[I])\}.$$

It readily follows from the definition that

$$[\kappa]^n \setminus F_n = \bigcup_{I \subsetneq n, j \in n \setminus I} S_{I,j}. \quad (11)$$

Let $I_0 \subsetneq n$ and $j_0 \in n \setminus I_0$ be such that

$$\mu_n(S_{I_0,j_0}) > 0.$$

Set $J = I_0 \cup \{j_0\}$, and let $\pi_J : [\kappa]^n \rightarrow [\kappa]^J$ be the canonical projection $\pi_J(s) = s[J]$. It follows that

$$\mu_m(\pi_J(S_J)) > 0,$$

²Indeed it suffices to assume that the sequence is weakly countably-null

where $m = |J|$. By Fubini's Theorem, the set of $t \in [\kappa]^{m-1}$ such that $\mu((\pi_J(S_J))_t) > 0$, has μ_{m-1} -positive measure, where $(\pi_J(S_J))_t := \{\alpha < \kappa : t \cup \{\alpha\} \in \pi_J(S_J)\}$. Observe that

$$(\pi_J(S_J))_t = \{\alpha < \kappa : \text{there is some } s \in S_J \text{ such that } s[I_0] = t \text{ and } (s)_{j_0} = \alpha\} \subseteq f(t),$$

so $(\pi_J(S_J))_t$ is countable, and hence $\mu(\pi_J(S_J))_t = 0$, a contradiction. \square

Claim 6.6.2. *Suppose that $s \in F_n$. Then $(x_\alpha)_{\alpha \in s}$ is a 1-unconditional basic sequence.*

Proof of Claim: Let $(a_\alpha)_{\alpha \in s}$ be a sequence of scalars and fix $t \subseteq s$. Let $f \in N_t$ be such that

$$f\left(\sum_{\alpha \in t} a_\alpha x_\alpha\right) = \left\| \sum_{\alpha \in t} a_\alpha x_\alpha \right\|. \quad (12)$$

Since s is θ -free, it follows that $f(t) \cap s \subseteq t$. This means that for every $\alpha \in s \setminus t$ one has that $f(x_\alpha) = 0$. So,

$$\left\| \sum_{\alpha \in s} a_\alpha x_\alpha \right\| \geq f\left(\sum_{\alpha \in s} a_\alpha x_\alpha\right) = f\left(\sum_{\alpha \in t} a_\alpha x_\alpha\right) = \left\| \sum_{\alpha \in t} a_\alpha x_\alpha \right\|. \quad (13)$$

\square

Recall that our assumption is that there is a non-trivial σ -additive real-valued probability measure μ on κ such that for every $n \in \mathbb{N}$, $t \in F_{n-1}$ and every $i < n$, the sets

$$A_t^i := \{\alpha < \kappa : \text{there is some } u \in F_n \text{ such that } u(i) = \alpha \text{ and } u[n \setminus \{i\}] = t\}$$

are all μ -measurable. So what we have accomplished is that the sets F_n are μ_n -measurable for every $n \in \mathbb{N}$. It follows from the Claim 6.6.1 and the Claim 6.6.2 that the sets $U_n := \{s \in [\kappa]^n : (x_\alpha)_{\alpha \in s} \text{ is 1-unconditional}\}$ have μ_n -measure 1 for every n . Let μ_ω denote the infinite product measure on κ^ω . Then it follows that the set of sequences $(\alpha_n)_{n \in \omega}$ such that $(x_{\alpha_n})_{n \in \omega}$ is 1-unconditional has measure 1, and we are done. \square

7. SUBSYMMETRIC SEQUENCES

Definition 7.1. Let A be a set of ordinal numbers and let $\mathcal{B} \subseteq [A]^{<\omega}$. The family \mathcal{B} is pre-compact when its topological closure consists only on finite sets, when we consider the topology product topology on 2^A . The family \mathcal{B} is compact when \mathcal{B} is closed with respect to the product topology on 2^A . The family \mathcal{B} is hereditary when $t \subseteq s \in \mathcal{B}$ implies that $t \in \mathcal{B}$. Recall that we say that the family \mathcal{B} is *large* in A when for every $B \in [A]^\omega$ and every $n \in \mathbb{N}$ we have that $\mathcal{B} \cap [B]^n \neq \emptyset$.

It is easy to see that if \mathcal{B} is hereditary, then \mathcal{B} is pre-compact if and only if \mathcal{B} is compact. It is also an exercise to prove that \mathcal{B} is pre-compact if and only if for every infinite sequence $(s_n)_{n \in \omega}$ in \mathcal{B} has a subsequence $(s_n)_{n \in B}$ forming a Δ -system, i.e. there is some $s \in [A]^{<\omega}$ such that for every $n \neq m$ in A one has that $s_n \cap s_m = s$.

Definition 7.2. Let κ be a cardinal number and $c : [\kappa]^{<\omega} \rightarrow X$. A subset H of κ is called *c-homogeneous* if for every $n < \omega$ the restriction $c \upharpoonright [H]^n$ is constant. Given such c , we define $\mathcal{H}(c)$ as the family of all *c-homogeneous* subsets of κ .

Recall that a cardinal number κ is called ω -Erdős if

$$\kappa \longrightarrow (\omega)_2^{<\omega},$$

i.e., for every $c : [\kappa]^{<\omega} \rightarrow 2$ the family $\mathcal{H}(c)$ consists only on finite subsets of κ .

In general, $\mathcal{H}(c)$ is a compact and hereditary family of $\mathcal{P}(\kappa)$. It is not difficult to see that κ is ω -Erdős if and only if every $c : [\kappa]^{<\omega} \rightarrow 2^{\aleph_0}$ there are no infinite c -homogeneous subsets of κ .

Proposition 7.3. *Let $c : [\kappa]^{<\omega} \rightarrow 2$. The family $\mathcal{H}(c)$ is a large compact and hereditary family of (not necessary finite) subsets of κ .*

PROOF. It is clear that $\mathcal{H}(c)$ is hereditary and closed in $\mathcal{P}(\kappa)$. Let us prove that $\mathcal{H}(c)$ is large. Given integers k, l, m , let $R(k, l, m)$ be the minimal integer such that whenever $d : [R(k, l, m)]^k \rightarrow m$ there is a subset $A \subseteq R(k, l, m)$ of cardinality l such that $d \upharpoonright [A]^k$ is constant.

Fix now an infinite subset $A \subseteq \kappa$ and $n < \omega$. Let s be a finite subset of A of cardinality $|s| \geq R(n, 2n-1, 2^n)$. Let $d : [s]^n \rightarrow 2^n$ be defined for $t \in [s]^n$ by $d(t) := (c(t[i]))_{i < n}$. Let $u \in [s]^{2n-1}$ be such that $c \upharpoonright [u]^n$ is constant, with value $(\varepsilon_i)_{i < n} \in 2^n$. Let $v := u[n]$. We claim that v is c -homogeneous: This is an easy consequence of the fact that for every $i < n$ and every $w \in [v]^i$ there is $z \in [u]^n$ such that $z[i] = w$. \square

Note that ω is not ω -Erdős: Let \mathcal{S} denote the Schreier family on ω . Then the characteristic function $\mathbb{1}_{\mathcal{S}}$ does not have infinite homogeneous sets: Suppose that there is some infinite $A \subseteq \omega$ such that $\mathbb{1}_{\mathcal{S}} \upharpoonright [A]^n$ is constant, with value $\varepsilon_n \in 2$ for every $n < \omega$. Since \mathcal{S} is hereditary, we get that

$$\varepsilon_n = \min_{m < n} \varepsilon_m \text{ for every } n < \omega. \quad (14)$$

This means that, either there is some m such that $\varepsilon_m = 0$, and hence $\mathcal{S} \cap [A]^m = \emptyset$ contradicting the fact that \mathcal{S} is large, or else $\varepsilon_m = 1$ for all m , hence $[A]^{<\omega} \subseteq \mathcal{S}$, contradicting the compactness of \mathcal{S} . Following this idea, the general situation is completely understood.

Proposition 7.4. *Let κ be an infinite cardinal. The following are equivalent:*

- (1) κ is ω -Erdős.
- (2) Every separated normalized sequence $(x_\alpha)_{\alpha < \kappa}$ has a subsymmetric subsequence.
- (3) There are no large compact and hereditary families on κ .

PROOF. (1) implies (2): This is due to [Ke] but for the convenience of the reader, we sketch the proof here. Suppose that κ is a ω -Erdős cardinal, and suppose that $(x_\alpha)_{\alpha < \kappa}$ is a normalized separated sequence. Let $c : [\kappa]^{<\omega} \rightarrow 2^{\aleph_0}$ be defined as follows. Let $s \in [\kappa]^{<\omega}$. Note that $\vartheta_{s,|s|}$ linearly extends to $\vartheta_{s,|s|} : \langle x_\gamma \rangle_{\gamma \in s} \rightarrow \langle u_n \rangle_{n \in |s|}$ by $\vartheta_{s,|s|}(u_\gamma) := u_{\vartheta_{s,|s|}(\gamma)}$ for each $\gamma \in s$. Let $c(s) := (|s|, (f \circ \vartheta_{s,|s|})_{f \in (\langle x_\gamma \rangle_{\gamma \in s})^*})$. Let $A \subseteq \kappa$ be an infinite set such that $c \upharpoonright [A]^n$ is constant for every n . This implies that $(x_\gamma)_{\gamma \in A}$ is 1-subsymmetric.

(2) implies (3): Suppose that \mathcal{B} is a large compact and hereditary family on κ . Now, we use the family \mathcal{B} to define the following Schreier norm in $c_{00}(\kappa)$: For $x \in c_{00}(\kappa)$ let

$$\|x\|_{\mathcal{B}} := \max\{|\langle x, \chi_s \rangle| : s \in \mathcal{B}\}.$$

Observe that $[\kappa]^1 \subseteq \mathcal{B}$, so the formula above defines a norm on $c_{00}(\kappa)$. Let $X_{\mathcal{B}}$ be the completion of $c_{00}(\kappa)$ with respect to the norm $\|\cdot\|_{\mathcal{B}}$. It readily follows from the fact that \mathcal{B} is hereditary, that the unit Hamel basis $(u_\gamma)_{\gamma < \kappa}$ is an unconditional Schauder basis of $X_{\mathcal{B}}$.

Claim 7.4.1. *$(u_\gamma)_{\gamma < \kappa}$ is a normalized weakly-null sequence without subsymmetric subsequences.*

Proof of Claim: Fix $A \in [\kappa]^\omega$. We first check that $(u_\gamma)_{\gamma \in A}$ is weakly-null. This is a standard fact: Suppose that there is some $f \in (X_{\mathcal{B}})^*$ and $\varepsilon > 0$ the set $B := \{\gamma \in A : f(u_\gamma) \geq \varepsilon\}$ is infinite. Now since $\mathcal{B} \upharpoonright B$ is a compact family on B , there is by *Ptak's Lemma*, some $\mu \in S_{\ell_1(B)}^+ \cap c_{00}(B)$ such that

$$|\langle \mu, \mathbb{1}_s \rangle| \leq \frac{\varepsilon}{2} \text{ for every } s \in \mathcal{B} \upharpoonright A. \quad (15)$$

Then,

$$\frac{\varepsilon}{2} \geq \left\| \sum_{\gamma \in \text{supp } \mu} (\mu)_\gamma u_\gamma \right\|_{\mathcal{B}} \geq f\left(\sum_{\gamma \in \text{supp } \mu} (\mu)_\gamma u_\gamma \right) \geq \varepsilon, \quad (16)$$

which is, of course, impossible.

Now we prove that $(u_\gamma)_{\gamma \in A}$ is not subsymmetric: Fix $C \geq 1$. Use again Ptak's Lemma to find $\mu \in S_{\ell_1(A)}^+ \cap c_{00}(A)$ such that $|\langle \mu, \chi_s \rangle| \leq C/2$ for every $s \in \mathcal{B} \upharpoonright A$. Let as above $x := \sum_{\gamma \in \text{supp } \mu} (\mu)_\gamma u_\gamma$. Then $\|x\|_{\mathcal{B}} \leq C/2$. Since \mathcal{B} is large, there is some $s \in \mathcal{B} \upharpoonright [A]^n$, where $n = |\text{supp } \mu|$. Let $y := \sum_{\gamma \in s} (\mu)_{\vartheta_{s, \text{supp } \mu}(\gamma)} u_\gamma$. Then

$$\|y\|_{\mathcal{B}} \geq \langle y, \chi_s \rangle = \sum_{\gamma \in \text{supp } \mu} (\mu)_\gamma = 1 > \frac{1}{C} \|x\|_{\mathcal{B}}. \quad (17)$$

□

(3) implies (1): Suppose that κ is not ω -Erdős. Fix a coloring $c : [\kappa]^{<\omega} \rightarrow 2$ without infinite c -homogeneous sets. Then $\mathcal{H}(c)$ is, by Proposition 7.3, a large compact and hereditary family of finite subsets of κ . □

8. NON-SEPARABLE SPACES OF TSIRELSON TYPE

We have seen that for every non ω -Erdős cardinal κ there is a large family \mathcal{B} on κ . For $\kappa = \omega$, these families can be used to provide a separable Banach space without subsymmetric sequences. A large family on ω was used to build the Tsirelson space³ which is a reflexive space with an unconditional basis and without subsymmetric basic sequences. Its construction is naturally generalized to an arbitrary infinite cardinal number κ . The goal of this subsection is to prove that non-separable Tsirelson spaces have always subsymmetric basic sequences, indeed isomorphic copies of some of the classical sequence spaces c_0 or ℓ_1 , $p \geq 1$.

Towards this goal, in all this subsection we fix an infinite cardinal number κ , an *hereditary* family \mathcal{B} of finite subsets of κ , and a real number $0 < \theta < 1$.

Definition 8.1. We say that a finite block sequence $(E_i)_{i < n}$ of finite subsets of κ is \mathcal{B} -admissible when there is $\{\gamma_i\}_{i < n} \in \mathcal{B}$ such that $\gamma_0 \leq E_0 < \gamma_1 \leq E_1 < \dots < \gamma_{n-1} \leq E_n$. Similarly, a finite sequence $(x_i)_{i=1}^n$ of vectors of $c_{00}(\kappa)$ is called \mathcal{B} -admissible when $(\text{supp } x_i)_{i=1}^n$ is \mathcal{B} -admissible.

Given such \mathcal{B} and a real number $0 < \theta < 1$, we define the *Tsirelson-like space* $T_{\theta, \mathcal{B}} := (T_{\theta, \mathcal{B}}, \|\cdot\|_{\theta, \mathcal{B}})$ on κ as follows: The norm $\|\cdot\|_{\theta, \mathcal{B}}$ is the unique norm on $c_{00}(\kappa)$ satisfying that for every $x \in c_{00}(\kappa)$ we have that

$$\|x\|_{\theta, \mathcal{B}} = \max\{\|x\|_\infty, \sup\{\theta \cdot \sum_{i < n} \|E_i x\|_{\theta, \mathcal{B}} : (E_i)_{i < n} \text{ is } \mathcal{B}\text{-admissible}\}\}. \quad (18)$$

Then $T_{\theta, \mathcal{B}}$ is the completion of the normed space $(c_{00}(\kappa), \|\cdot\|_{\theta, \mathcal{B}})$.

³indeed, its dual

Since the family \mathcal{B} is hereditary, it follows easily that

$$\text{if } (E_i)_{i < n} \text{ is } \mathcal{B}\text{-admissible, and } F_i \subseteq E_i, i < n, \text{ then so is } (F_i)_{i < n}. \quad (19)$$

This fact readily implies that the unit basis $(u_\gamma)_{\gamma < \kappa}$ is a 1-unconditional Schauder basis of the Tsirelson-like space $T_{\theta, \mathcal{B}}$.

The classical Tsirelson example is $T := T_{1/2, \mathcal{S}}$, where \mathcal{S} is the Schreier family on ω . An interesting result of Bellenot [Be] states that $T_{\theta, [\kappa] \leq n}$ is isomorphic to $c_0(\kappa)$, if $\theta n \leq 1$ or to $\ell_p(\kappa)$ for $p = \log(n)/(\log(n) + \log(\theta))$. In general, the situation is well understood for a compact family \mathcal{B} in ω : If the Cantor-Bendixson rank of \mathcal{B} is an integer n , then the space $T_{\theta, \mathcal{B}}$ is saturated by copies of either c_0 when $\theta n \leq 1$, or by copies of ℓ_p , with $p = \log(n)/(\log(n) + \log(\theta)) > 1$ if $\theta n > 1$. In this latter case, the space $T_{\theta, \mathcal{B}}$ is therefore, by a classical result of James, reflexive (see [Be-De]). If otherwise the Cantor-Bendixson rank of \mathcal{B} is infinite, then the space $T_{\theta, \mathcal{B}}$ does not contain subsymmetric sequences, and therefore reflexive (see for example [Lo-Ma]). In particular, $T_{\theta, \mathcal{B}}$ never contains a copy of ℓ_1 . If $\kappa > \omega$, the situation is very much different. To visualize this difference easily, let us assume that κ is not a cardinal but the ordinal number ω^2 , and let \mathcal{B} be the family

$$\mathcal{B} := \{\emptyset\} \cup \{\{\omega \cdot i + n\}_{i < m} : n < \omega, \text{ and } m \leq n\}.$$

Then the only accumulation point in \mathcal{B} is \emptyset , and hence its Cantor-Bendixson rank is 1. However every sequence $(u_\gamma)_{\gamma \in s}$ with $s \subseteq \{\omega \cdot n\}_{n > 0}$ is \mathcal{B} -admissible, and hence $(u_{\omega \cdot n})_{n > 0}$ is θ^{-1} -equivalent to the unit basis of ℓ_1 . It is easy to modify \mathcal{B} to make it with infinite Cantor-Bendixson rank and still having a copy of ℓ_1 .

The main result is the following.

Theorem 8.2.

- (1) *Suppose that κ is an uncountable regular cardinal. Then there is a club C of κ such that either*
 - (1.1) *the sequence $(u_\gamma)_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ is θ^{-1} -equivalent to the unit basis of $\ell_1(\kappa)$ or*
 - (1.2) *the closed linear span of $\langle u_\gamma \rangle_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ is either c_0 -saturated or ℓ_p -saturated for some $p \geq 1$.*
- (2) *Suppose that κ is an uncountable singular cardinal. Then for every cardinal number $\lambda < \kappa$ there is a subset C_λ of κ of cardinality λ such that either*
 - (1.1) *the sequence $(u_\gamma)_{\gamma \in C_\lambda}$ in $T_{\theta, \mathcal{B}}$ is θ^{-1} -equivalent to the unit basis of $\ell_1(\lambda)$ or*
 - (1.2) *the closed linear span of $\langle u_\gamma \rangle_{\gamma \in C_\lambda}$ in $T_{\theta, \mathcal{B}}$ is either c_0 -saturated or ℓ_p -saturated for some $p \geq 1$.*

Definition 8.3. Let $n \in \mathbb{N}$. The (\mathcal{B}, n) -Namba game, or simply the (\mathcal{B}, n) -game $\mathfrak{D}_{\mathcal{B}, n}$ is the following game of height n : The first Player I starts with an ordinal $\gamma_0 < \kappa$, and the Player II replies with an ordinal $\gamma_0 < \eta_0 < \kappa$. Then the Player I replies with $\eta_0 < \gamma_1 < \kappa$, and then Player II chooses $\gamma_1 < \eta_1 < \kappa$, and so on. The game ends after n runs of each player. Player I wins if the set $\{\gamma_i\}_{i < n} \in \mathcal{B}$; otherwise II wins.

Definition 8.4. A strategy for Player I or Player II in the game $\mathfrak{D}_{\mathcal{B}, n}$ is a mapping $\sigma : [\kappa]^{< n-1} \rightarrow \kappa$ such that $s < \sigma(s)$ for every $s \in [\kappa]^{< n-1}$. An strategy σ for Player I is *winning* for the game

$\mathcal{D}_{\mathcal{B},n}$ if $\{\sigma(\emptyset), \sigma(\{\eta_0\}), \sigma(\{\eta_0, \eta_1\}), \dots, \sigma(\{\eta_0, \eta_1, \dots, \eta_{n-2}\})\}$ is in \mathcal{B} for every $\eta_0 < \eta_1 < \dots < \eta_{n-2} < \kappa$. A strategy σ for Player *II* is *winning* for the game $\mathcal{D}_{\mathcal{B},n}$ if $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ is not in \mathcal{B} for every $\gamma_0 < \gamma_1 < \dots < \gamma_{n-1} < \kappa$ such that $\gamma_0 < \sigma(\{\gamma_0\}) < \gamma_1 < \sigma(\{\gamma_0, \gamma_1\}) < \dots < \gamma_{n-2} < \sigma(\{\gamma_0, \dots, \gamma_{n-2}\}) < \gamma_{n-1}$.

Few useful facts.

Proposition 8.5. (a) *The game $\mathcal{D}_{\mathcal{B},n}$ is determined, i.e., either Player *I* has a winning strategy or Player *II* has a winning strategy.*
 (b) *If player *I* has a winning strategy for the game $\mathcal{D}_{\mathcal{B},n}$ then he also has a winning strategy for the game $\mathcal{D}_{\mathcal{B},m}$ for every $m \leq n$. Symmetrically, if player *II* has a winning strategy for the game $\mathcal{D}_{\mathcal{B},n}$ then he also has a winning strategy for the game $\mathcal{D}_{\mathcal{B},m}$ for every $m \geq n$.*

PROOF. (a) is a consequence of the fact that the game $\mathcal{D}_{\mathcal{B},n}$ is finite, (b) and (c) are consequences of the fact that \mathcal{B} is hereditary. \square

Definition 8.6. A set $C \subseteq \kappa$ is closed under a strategy σ of Player *I* or Player *II* for the game $\mathcal{D}_{\mathcal{B},n}$ when $\sigma(s) < C/\gamma$ for every $\gamma \in C$ and $s \in [\gamma + 1]^{<n-1}$.

Proposition 8.7. *Suppose that κ is an uncountable regular cardinal. Then for every strategy σ of Player *I* or Player *II* there is a club $C \subseteq \kappa$ closed under σ .*

PROOF. Fix $\sigma : [\kappa]^{<n-1} \rightarrow \kappa$. We define $C := \{\gamma_\xi\}_{\xi < \kappa}$ inductively: If ξ is a limit ordinal, then $\gamma_\xi := \sup_{\eta < \xi} \gamma_\eta$. For the successor case $\xi + 1$, Let

$$\gamma_{\xi+1} := \sup\{\sigma(s) : s \in [\gamma_\xi + 1]^{<n-1}\} + \omega.$$

Since κ is uncountable and regular $\gamma_\xi < \gamma_{\xi+1} < \kappa$. It is easy to see that C has the required properties. \square

Definition 8.8. Given a family \mathcal{B} of finite subsets of κ , let

$$\alpha(\mathcal{B}) := \sup\{n < \omega : \text{Player } I \text{ has a winning strategy for the game } \mathcal{D}_{\mathcal{B},n}\}.$$

Definition 8.9. Let $C \subseteq \kappa$.

- (a) C is (\mathcal{B}, I) -closed when for every integer $n \leq \alpha(\mathcal{B})$ C is closed under a winning strategy for the game $\mathcal{D}_{\mathcal{B},n}$.
- (b) C is (\mathcal{B}, II) -closed when C is closed under a winning strategy of Player *II* in the game $\mathcal{D}_{\mathcal{B}, \alpha(\mathcal{B})+1}$, if $\alpha(\mathcal{B})$ is an integer.
- (c) C is \mathcal{B} -strategically closed when it is (\mathcal{B}, I) -closed and (\mathcal{B}, II) -closed.

Proposition 8.10. (a) *The three notions above are hereditary.*

- (b) *Suppose that κ is an uncountable regular cardinal. Then there is a club C of κ which is \mathcal{B} -strategically closed.*

PROOF. The first part is trivial. The second one is an immediate consequence of Proposition 8.7 and the fact that a countable intersection of clubs of κ is also a club of κ . \square

Recall that our main goal is to find a large subset C of κ and identify the closed subspace of $T_{\theta, \mathcal{B}}$ spanned by $\{u_\gamma\}_{\gamma \in C}$. This would be relatively easy to do if the sequence $(u_\gamma)_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ were equivalent to itself as a sequence in the space $T_{\theta, \mathcal{B} \upharpoonright C}$, because we would have reduced the main work to study restrictions of the family \mathcal{B} . However this is not the case, but there is another family \mathcal{F} of finite subsets of C for which the sequence $(u_\gamma)_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ and in $T_{\theta, \mathcal{F}}$ are 1-equivalent (see Proposition 8.13).

Definition 8.11. Given $\Gamma \subseteq \kappa$, $\Gamma \neq \emptyset$ and without maximum, let $\varpi_C : \sup C \rightarrow C$ be defined by $\varpi(\gamma) = \min\{\delta \in C : \delta \geq \gamma\}$. It is clear that ϖ_C is onto. Given a family \mathcal{B} of finite subsets of κ , let

$$\varpi_\Gamma(\mathcal{B}) := \{\varpi_\Gamma''(s) : s \in \mathcal{B}\}.$$

Let also

$$\Gamma^+ := \{\gamma \in \Gamma : \text{there is some } \eta \in \Gamma \text{ such that }]\delta, \gamma] = \{\gamma\}\}.$$

In other words, Γ^+ is the set of successors of Γ .

Observe that for $\Gamma_0 \subseteq \Gamma_1$, the mappings ϖ_{Γ_0} and $\varpi_{\Gamma_1} \upharpoonright \sup \Gamma_0$ are in general different. However, it is easy to see that if Γ_0 is an interval of Γ_1 , then the corresponding two mappings coincide. It is also clear from the definition that the family $\varpi_\Gamma(\mathcal{B})$ is hereditary because so is \mathcal{B} , and that $\mathcal{B} \upharpoonright C \subseteq \varpi_C(\mathcal{B})$.

The following two are useful facts.

Proposition 8.12. *Suppose that $(s_i)_{i < d}$ is a block sequence of subsets of Γ with $\{\min s_i\}_{i < d} \in \varpi_\Gamma(\mathcal{B})$. Then the sequence $(s_i)_{i < d}$ is \mathcal{B} -admissible.*

PROOF. Let $\{\delta_0 < \dots < \delta_{d-1}\} \in \mathcal{B}$ be such that $\min s_i = \varpi_\Gamma(\delta_i)$ for $i < d$. Then, by definition of ϖ_Γ and the fact that $\varpi_\Gamma(\delta_{i+1}) = \gamma_{i+1} > \max s_i$, we have that $\gamma_i < \delta_{i+1}$ for every $i < d-1$. Hence, $\delta_0 \leq \gamma_0 \leq \max s_0 < \delta_1 \leq \gamma_1 \leq \max s_1 < \dots < \delta_{d-1} \leq \gamma_{d-1}$ and consequently $(s_i)_{i < d}$ is \mathcal{B} -admissible. \square

Proposition 8.13. *Let $x \in c_{00}(\Gamma)$. Then $\|x\|_{\theta, \mathcal{B}} = \|x\|_{\theta, \varpi_\Gamma(\mathcal{B})}$.*

PROOF. The proof is by induction on the cardinality k of the support of x . If $k = 1$, the result is trivial. The case $k > 1$ readily follows from the following.

Claim 8.13.1. *Let $(E_i)_{i < d}$ be a block sequence of finite subsets of Γ . Then $(E_i)_{i < d}$ is \mathcal{B} -admissible if and only if $(E_i)_{i < d}$ is $\varpi_\Gamma(\mathcal{B})$ -admissible.*

Proof of Claim: Suppose that $(E_i)_{i < d}$ is \mathcal{B} -admissible. Let $\{\eta_i\}_{i < d} \in \mathcal{B}$ be such that $\eta_0 \leq E_0 < \eta_1 \leq E_1 < \dots < \eta_{d-1} \leq E_{d-1}$. Then $\varpi_\Gamma(\eta_0) \leq E_0 < \varpi_\Gamma(\eta_1) \leq E_1 < \dots < \varpi_\Gamma(\eta_{d-1}) < E_{d-1}$. Since $\{\varpi_\Gamma(\eta_i)\}_{i < d} \in \varpi_\Gamma(\mathcal{B})$, it follows that $(E_i)_{i < d}$ is also \mathcal{F} -admissible.

Suppose now that $(E_i)_{i < d}$ is $\varpi_\Gamma(\mathcal{B})$ -admissible. Let $\{\gamma_i\}_{i < d} \in \varpi_\Gamma(\mathcal{B})$ be such that $\gamma_0 \leq E_0 < \gamma_1 \leq E_1 < \dots < \gamma_{d-1} \leq E_{d-1}$, and let $\{\eta_i\}_{i < d} \in \mathcal{B}$ be such that $\gamma_i := \varpi_\Gamma(\eta_i)$, $i < d$. Since $\max E_i < \gamma_i \leq \min E_{i+1}$, $i < d-1$, it follows that $\max E_i < \eta_i \leq \min E_{i+1}$, $i < d-1$, so $(E_i)_{i < d}$ is \mathcal{B} -admissible. \square

\square

Recall that the Cantor-Bendixson index $\varrho_{\text{CB}}(\mathcal{F})$ of a *compact* family \mathcal{F} of finite subsets of κ is its Cantor-Bendixson index of the family viewed as a closed subset of 2^κ . Since \mathcal{F} is a scattered compactum, there is an ordinal α such that the α^{th} -derivative of \mathcal{F} is empty. Thus, $\varrho_{\text{CB}}(\mathcal{F})$ is the first α such that the $\alpha + 1$ -derivative of \mathcal{F} is empty.

Proposition 8.14. *Suppose that \mathcal{F} is a compact and hereditary family of subsets of some infinite set $C \subseteq \kappa$.*

- (a) *If there is $\gamma \in C$ and some infinite subset $A \subseteq C \setminus \{\gamma\}$ such that $\{\{\gamma\} \cup s : s \in [A]^{n-1}\} \subseteq \mathcal{F}$, then $\varphi_{\text{CB}}(\mathcal{B}) \geq n$.*
- (b) *If $\varrho_{\text{CB}}(\mathcal{F}) \geq n$, then there is $\{\gamma_0 < \dots < \gamma_{n-1}\}$ in \mathcal{F} such that $\gamma_i, \gamma_{i+1}[\cap C \neq \emptyset$ for every $i < n - 1$.*

PROOF. (a): We prove that $\{\gamma\} \in \partial^{(n)}\mathcal{F}$ by induction on n . Suppose that $\{\{\gamma\} \cup s : s \in [A]^n\} \subseteq \mathcal{F}$. Then $\{\{\gamma\} \cup s : s \in [A]^{n-1}\} \subseteq \partial(\mathcal{F})$, so by inductive hypothesis, $\{\gamma\} \in \partial^{(n-1)}(\partial\mathcal{F}) = \partial^{(n)}(\mathcal{F})$, and we are done.

(b): Induction on n . Suppose that $\varrho_{\text{CB}}(\mathcal{F}) \geq n + 1$. Since $\varrho_{\text{CB}}(\partial\mathcal{F}) \geq n$, there is $s = \{\gamma_0 < \dots < \gamma_{n-1}\} \in \partial\mathcal{F}$ such that $\gamma_i, \gamma_{i+1}[\cap C \neq \emptyset$, $i < n - 1$. Since $s \in \partial\mathcal{F}$ there is a sequence $(s_k)_{k \in \mathbb{N}}$ in $\mathcal{F} \setminus \{s\}$ with limit s . Since \mathcal{F} is hereditary, we may assume that $s_k = s \cup \{\eta_k\}$ for some $\eta_k \notin s$, $k \in \mathbb{N}$. Suppose first that there are k_0, k_1 such that $\eta_{k_0} < \eta_{k_1} < \gamma_0$. Then $\{\eta_{k_0}\} \cup s$ is the desired set. Suppose now there are k_0, k_1 such that $\gamma_{n-1} < \eta_{k_0} < \eta_{k_1}$. Then $\{\eta_{k_1}\} \cup s$ is the desired set. Otherwise, there is $i < n - 1$ and there are k_0, k_1, k_2 such that $\gamma_i < \eta_{k_0} < \eta_{k_1} < \eta_{k_2} < \gamma_{i+1}$, and then $s \cup \{\eta_{k_1}\}$ is the desired set. \square

Lemma 8.15. *Suppose that Player I has a winning strategy σ for the game $\mathfrak{D}_{\mathcal{B}, n}$. Let $C \subseteq \kappa$ be a set closed under σ and unbounded in itself. Suppose that $\gamma_1 < \dots < \gamma_{n-1}$ are successor elements of C . Then $\{\min C, \gamma_1, \dots, \gamma_{n-1}\} \in \varpi_C(\mathcal{B})$. Consequently,*

- (a) *$\{s \in [C]^n : \min s = \min C \text{ and } s \setminus \{\min s\} \subseteq C^+\} \subseteq \varpi_C(\mathcal{B})$, and $\varrho_{\text{CB}}(\mathcal{B}) \geq n$, if \mathcal{B} is compact.*
- (b) *$[C^+]^{n-1} \subseteq \varpi_C(\mathcal{B})$, and*
- (c) *every block sequence $(s_i)_{i < d}$ of finite subsets of C of length $d \leq n$ is \mathcal{B} -admissible.*

PROOF. Suppose that $\{\gamma_1 < \dots < \gamma_{n-1}\} \subseteq C^+$, and for $0 < i < n$, let $\delta_i \in C$ be such that $[\delta_i, \gamma_i] = \{\gamma_i\}$. We play the following run in the game $\mathfrak{D}_{\mathcal{B}, n}$: Player I plays $\sigma(\emptyset)$. Since C is closed under σ , it follows that $\sigma(\emptyset) < \delta_1$. Now let Player II play δ_1 . Next, Player I plays $\delta_1 < \sigma(\delta_1)$. It follows then that $\sigma(\delta_1) < \gamma_1 \leq \delta_2$, so let Player II play δ_2 , and so on. At the end of the game we see that $\delta_i < \sigma(\delta_1, \dots, \delta_i) < \gamma_i$ for every $i = 1, \dots, n - 1$, and since $[\delta_i, \gamma_i] = \{\gamma_i\}$, it follows that $\varpi_C(\sigma(\delta_1, \dots, \delta_i)) = \gamma_i$ for every $i = 1, \dots, n$. Since σ is winning, $\{\sigma(\emptyset), \sigma(\delta_1), \dots, \sigma(\delta_1, \dots, \delta_{n-1})\} \in \mathcal{B}$. But $\{\sigma(\emptyset), \sigma(\delta_1), \dots, \sigma(\delta_1, \dots, \delta_{n-1})\} = \{\min C, \gamma_1, \dots, \gamma_{n-1}\}$, and we are done.

(a) Follows from the previous and Proposition 8.14 (a). (b) is a trivial consequence of (a). Let us prove (c): Let $(s_i)_{i < n}$ be a block sequence of finite subsets of C . By adding convenient ordinal numbers to the sets s_i , $i < n$ we may assume wlog that $\min s_0 = \min C$, and that $\min s_i$ are not limit elements of C . Hence $\{\min s_i\}_{i < n} \in \varpi_C(\mathcal{B})$, and so, by Proposition 8.12, $(s_i)_{i < n}$ is \mathcal{B} -admissible. \square

Lemma 8.16. *Suppose that Player II has a winning strategy σ for the game $\mathfrak{D}_{\mathcal{B},n}$. Let $C \subseteq \kappa$ be a set closed under σ which is unbounded in itself. Suppose that $\gamma_0 < \dots < \gamma_{n-1}$ are in C and are such that $]\gamma_i, \gamma_{i+1}[\cap C \neq \emptyset$ for every $i < n-1$. Then $\{\gamma_i\}_{i < n} \notin \varpi_C(\mathcal{B})$. Consequently,*

(a) *If $D \subseteq C$ is such that $]\gamma, \eta[\cap C \neq \emptyset$ for every $\gamma < \eta$ in D , then $[D]^n \cap \varpi_C(\mathcal{B}) = \emptyset$.*

(b) *$\varpi_C(\mathcal{B}) \subseteq [C]^{<2n-1}$, and if $(s_i)_{i < d}$ is a \mathcal{B} -admissible sequence of finite subsets of C then*

$$d < 2n - 1 \text{ and } |\{i < d : |s_i| \geq 2\}| \leq n. \quad (20)$$

(c) *$\varpi_C(\mathcal{B})$ is compact and $\varrho_{CB}(\mathcal{B}) \leq n$.*

PROOF. Let $\gamma_0 < \dots < \gamma_{n-1}$ be as in the hypothesis of the Lemma, and suppose otherwise that $\{\gamma_i\}_{i < n} \in \varpi_C(\mathcal{B})$. Fix then $\{\eta_0 < \dots < \eta_{n-1}\} \in \mathcal{B}$ such that $\gamma_i = \varpi_C(\eta_i)$, $i < n$ and let $\delta_i \in C$ be such that $\delta_i \in]\gamma_i, \gamma_{i+1}[$, $i < n-1$. We play the following run in the game $\mathfrak{D}_{\mathcal{B},n+1}$: Player I plays η_0 . Then Player II replies with $\eta_0 < \sigma(\eta_0)$. Since C is closed under σ , and since $\eta_0 \leq \gamma_0$, it follows that $\sigma(\eta_0) < \delta_0 < \gamma_1$. Since $\varpi_C(\eta_1) = \gamma_1$, it follows that $\delta_0 < \eta_1$, and hence Player I can play η_1 , and Player II replies $\sigma(\eta_0, \eta_1)$. By a similar argument, $\sigma(\eta_0, \eta_1) < \eta_2$, so Player I can play η_2 , and so on. In this way Player I was able to produce $\{\eta_i\}_{i < n} \in \mathcal{B}$ against Player II following its winning strategy σ , a contradiction.

(a) is an easy consequence of the previous. Let us prove (b): Suppose otherwise that $\{\gamma_0 < \dots < \gamma_{2n-2}\} \in \varpi_C(\mathcal{B})$. Then $\{\gamma_{2i}\}_{i < n} \in \varpi_C(\mathcal{B})$ and $\gamma_{2i+1} \in]\gamma_{2i}, \gamma_{2(i+1)}[$, contradicting the statement in the Lemma. Suppose now that $(s_i)_{i < d}$ is a \mathcal{B} -admissible sequence of subsets of C , and suppose that $d \geq 2n-1$. Wlog we assume that $d = 2n-1$. Let $\{\eta_i\}_{i < 2n-1} \in \mathcal{B}$ be such that $\eta_0 \leq s_0 < \eta_1 \leq s_1 < \dots < \eta_{2n-2} \leq s_{2n-2}$. Then $\{\varpi_C(\eta_i)\}_{i < 2n-2} \in \varpi_C(\mathcal{B}) \cap [C]^{2n-1}$, contradicting the first part of (b). Similarly one proves that $|\{i < d : |s_i| \geq 2\}| \leq n$.

(c): Since $\varpi_C(\mathcal{B}) \subseteq [C]^{<2n-1}$, and $\varpi_C(\mathcal{B})$ is hereditary, the family $\varpi_C(\mathcal{B})$ is compact. Now $\varrho_{CB}(\varpi_C(\mathcal{B})) \leq n$ follows from the main statement in the Lemma, and Proposition 8.14. \square

Lemma 8.17. *Suppose that C is \mathcal{B} -strategically closed. Then*

(a) *$\varpi_C(\mathcal{B}) \upharpoonright C_0 = [C_0]^{<\omega}$ if $\alpha(\mathcal{B}) = \omega$, and where $C_0 = C^+ \cup \{\min C\}$.*

(b) *$\varrho_{CB}(\varpi_C(\mathcal{B})) = \alpha(\mathcal{B})$, when $\alpha(\mathcal{B}) < \omega$.*

PROOF. (a) readily follows from Lemma 8.15. Let us prove (b): From Lemma 8.16 (c) we know that $\varpi_C(\mathcal{B}) \subseteq [C]^{<2n-1}$, so $\varpi_C(\mathcal{B})$ is compact. From Lemma 8.15 (a) we obtain that $\varrho_{CB}(\varpi_C(\mathcal{B})) \geq \alpha(\mathcal{B})$. And $\varrho_{CB}(\varpi_C(\mathcal{B})) \leq \alpha(\mathcal{B})$ is a consequence of Lemma 8.16 (c). \square

Now Theorem 8.2 is follows readily from the more informative result.

Theorem 8.18. *Suppose that κ is an uncountable regular cardinal, and suppose that C is a club in κ which is \mathcal{B} -strategically closed. Then*

- (1) *the sequence $(u_\gamma)_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ is θ^{-1} -equivalent to the unit basis of $\ell_1(\kappa)$, if $\alpha(\mathcal{B}) = \omega$, or*
- (2) *$\alpha(\mathcal{B}) < \omega$ and the closed linear span of $\langle u_\gamma \rangle_{\gamma \in C}$ in $T_{\theta, \mathcal{B}}$ is c_0 -saturated if $\theta \cdot \alpha(\mathcal{B}) \leq 1$ or ℓ_p -saturated if $\theta \cdot \alpha(\mathcal{B}) > 1$ and where $p = \log(\alpha(\mathcal{B})) / (\log(\alpha(\mathcal{B})) + \log(\theta))$.*

PROOF. Let C be a club of κ which is \mathcal{B} -strategically closed. Suppose first that $\alpha(\mathcal{B}) = \omega$. Then from Lemma 8.17 (a) we obtain that every finite block sequence $(s_i)_{i < d}$ of finite subsets of C is \mathcal{B} -admissible, so it readily follows that $\|\sum_{\gamma \in s} a_\gamma u_\gamma\|_{\theta, \mathcal{B}} \geq \theta \sum_{\gamma \in s} |a_\gamma|$ for every $s \subseteq C$.

Suppose now that $\alpha(\mathcal{B}) < \omega$. Set $X_C := \overline{\langle u_\gamma \rangle_{\gamma \in C}}$. Let X be an infinite dimensional closed subspace of X_C . Since $(u_\gamma)_{\gamma < \kappa}$ is a Schauder basis of $T_{\theta, \mathcal{B}}$, we may assume that X is the closed linear span of a normalized block sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \langle u_\gamma \rangle_{\gamma \in C}$, for every $n \in \mathbb{N}$. Let $D := \bigcup_{n \in \mathbb{N}} \text{supp } x_n$. Then D is unbounded in D and it is \mathcal{B} -strategically closed. It follows that $\varrho_{\text{CB}}(\varpi_D(\mathcal{B})) = \alpha(\mathcal{B})$. It is proved in [Be-De] (see also [Lo-Ma]) that the separable Tsirelson-like space $T_{\theta, \varpi_D(\mathcal{B})}$ is saturated by copies of c_0 if $\theta \cdot \alpha(\mathcal{B}) \leq 1$, or by copies of ℓ_p if $\theta \cdot \alpha(\mathcal{B}) > 1$, with $p = \log(\alpha(\mathcal{B})) / (\log(\theta) + \log(\alpha(\mathcal{B})))$. But by Proposition 8.13, X_C and $T_{\theta, \varpi_D(\mathcal{B})}$ are the same space, so we are done. \square

Corollary 8.19. *Suppose that \mathcal{B} is a large hereditary family on an uncountable regular cardinal κ . Then $\alpha(\mathcal{B}) = \omega$ and consequently, the space $T_{\theta, \mathcal{B}}$ contains a copy of $\ell_1(\kappa)$.*

PROOF. Let C be unbounded in itself and \mathcal{B} -strategically closed. Since $[C]^n \cap \mathcal{B} \neq \emptyset$, for every n , we obtain $[C]^n \cap \varpi_C(\mathcal{B}) \neq \emptyset$. So, by Lemma 8.17, $\alpha(\mathcal{B}) = \omega$ and the desired result follows from Theorem 8.18. \square

A well-known procedure in producing compact and hereditary families in ω is given by the following operation: Given two families \mathcal{B} and \mathcal{C} in ω , we define

$$\mathcal{B} \otimes \mathcal{C} := \left\{ \bigcup_{i < d} s_i : (s_i)_{i < d} \text{ is a block sequence of elements of } \mathcal{B} \text{ such that } \{\min s_i\}_{i < d} \in \mathcal{C} \right\}.$$

It is not difficult to see that $\mathcal{B} \otimes \mathcal{C}$ is compact if both \mathcal{B} and \mathcal{C} are compact, and indeed $\varrho_{\text{CB}}(\mathcal{B} \otimes \mathcal{C}) = \varrho_{\text{CB}}(\mathcal{B}) \cdot \varrho_{\text{CB}}(\mathcal{C})$. In particular, $[\omega]^{\leq 2} \otimes \mathcal{S}$ is a compact family with Cantor-Bendixson rank ω , so comparable to \mathcal{S} . Similarly, we can produce the n -powers of \mathcal{S} , $\mathcal{S}_n := \mathcal{S} \otimes \dots \otimes \mathcal{S}$ with $\varrho_{\text{CB}}(\mathcal{S}_n) = \omega^n$. In the uncountable case, the situation is again very much different.

Proposition 8.20. *Suppose that \mathcal{B} is a large family on an uncountable regular cardinal κ . Then there is some subset Γ of κ of cardinality κ such that $[\Gamma]^\omega \subseteq [\kappa]^{\leq 2} \otimes \mathcal{B}$.*

PROOF. Let C be a club of κ which is \mathcal{B} -strategically closed, and let A be an arbitrary subset of C of cardinality κ such that $\min A \cap C$ and $]\gamma, \eta[\cap C$ are infinite sets for every $\gamma < \eta$ in A . We claim that $[A]^\omega \subseteq \mathcal{B}^2$: Let $B \in [A]^\omega$, $\gamma_i := \theta_{\omega, B}(i)$ for every $i < \omega$, and for each $i < \omega$, let $\{\delta_j^i\}_{j < \omega} \subseteq C$ be such that $\delta_0^0 < \delta_1^0 < \dots < \delta_j^0 < \gamma_0$, and $\gamma_i < \delta_0^{i+1} < \delta_1^{i+1} < \dots < \delta_j^{i+1} < \dots < \gamma_{i+1}$ for every $i < \omega$. Fix $n < \omega$. Since we have seen in Corollary 8.19 that $\alpha(\mathcal{B}) = \omega$, Lemma 8.17, implies that

$$\{\delta_n^0, \gamma_0, \delta_n^1, \gamma_1, \dots, \delta_n^n, \gamma_n\} \in \varpi_C(\mathcal{B}). \quad (21)$$

So there is $\{\eta_i\}_{i < \leq n} \in \mathcal{B}$ such that

$$\delta_n^0 < \eta_0 \leq \gamma_0 < \delta_n^1 < \eta_1 < \gamma_1 < \dots < \delta_n^i < \eta_i < \gamma_i < \dots < \delta_n^n < \eta_n < \gamma_n. \quad (22)$$

Then $s_n := \{\eta_0, \gamma_0, \eta_1, \gamma_1, \dots, \eta_n, \gamma_n\} \in [\kappa]^{\leq 2} \otimes \mathcal{B}$, and $s_n \rightarrow_n \{\gamma_0, \dots, \gamma_n\}$. Since n is arbitrary, this implies that B is in the topological closure of $[\kappa]^{\leq 2} \otimes \mathcal{B}^2$. \square

We shall need the following classical result (see, for example, [Wi]).

Theorem. ⁴[Erdős-Dushnik-Miller] For every colouring $c : [\kappa]^2 \rightarrow 2$ either there is $A \in [\kappa]^\kappa$ such that $c \restriction [A]^2$ is constant with value 0, or there is $B \in [\kappa]^\omega$ such that $c \restriction [B]^2$ is constant with value 1.

Proposition 8.21. Suppose that \mathcal{B} is a large family on an uncountable cardinal number κ . Then \mathcal{B}^2 is not compact. Indeed,

1. If κ is regular, there is a subset A of κ of cardinality κ such that $[A]^\omega \subseteq \overline{\mathcal{B}^2}$.
2. If κ is singular, for every cardinal number $\lambda < \kappa$ there is a subset A_λ of κ of cardinality λ such that $[A_\lambda]^\omega \subseteq \overline{\mathcal{B}^2}$.

PROOF. The singular case is a direct consequence of the regular case. So, we assume that κ is an uncountable regular cardinal number. Let $c : [\kappa]^2 \rightarrow 2$ defined for $s \in [\kappa]^2$ by $c(s) = 0$ iff $s \in \mathcal{B}$. By the Erdős-Dushnik-Miller theorem there must be a subset $A \subseteq \kappa$ of cardinality κ such that $[A]^2 \subseteq \mathcal{B}$, since the other possibility would imply that \mathcal{B} is not large. Hence $[A]^\omega \subseteq \mathcal{B}$, and the desired result follows from Proposition 8.20. \square

REMARK 8.22. Recall that a family \mathcal{B} is called *spreading* when if $(\{\gamma_i\}_{i < d})$ is \mathcal{B} -admissible, then $\{\gamma_i\}_{i < d} \in \mathcal{B}$. In particular, $\varpi_C(\mathcal{B}) = \mathcal{B} \restriction C$ for every $C \subseteq \kappa$ unbounded in C . So, if \mathcal{B} is a hereditary and spreading family on an uncountable cardinal κ , then there is an uncountable subset C of κ and some ordinal $\alpha \leq \omega$ such that $\mathcal{B} \restriction C = [C]^{<\alpha}$. This is basically consequence of Lemma 8.15 and Lemma 8.17.

REMARK 8.23. It is a well-known fact that the separable Tsirelson space T does not contain isomorphic copies of c_0 or ℓ_p , $p \geq 1$. Indeed, T does not have subsymmetric basic sequences. One may ask if there is some cardinal number κ such that every Banach space of such density κ contains an isomorphic copy of either c_0 or ℓ_p , $p \geq 1$. This is not true, as the following simple remark shows: Let X be the separable space constructed by Figiel and Johnson [Fi-Jo] with a subsymmetric⁵ basis $(v_n)_{n \in \omega}$ and not containing c_0 or ℓ_p , $p \geq 1$. Then define on $c_{00}(\kappa)$ the following norm: For $x \in c_{00}(\kappa)$, let

$$\|x\|_{X,\kappa} := \|\theta_{\text{supp } x, |\text{supp } x|}(x)\|_X \quad (23)$$

where $\theta_{\text{supp } x, |\text{supp } x|}(x) = \sum_{\gamma \in \text{supp } x} (x)_\gamma v_{\theta_{\text{supp } x, |\text{supp } x|}(\gamma)}$. The formula in (23) defines a norm on $c_{00}(\kappa)$ because $(v_n)_n$ is subsymmetric. Moreover, $(u_\gamma)_{\gamma < \kappa}$ is a subsymmetric basis of the completion $\mathfrak{X}_{X,\kappa}$ of $(c_{00}(\kappa), \|\cdot\|_{X,\kappa})$. A standard gliding-hump argument shows that $\mathfrak{X}_{X,\kappa}$ does not contain c_0 or ℓ_p , $p \geq 1$.

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⁴Using the Erdős-Rado arrow notation, this result is stated as $\kappa \rightarrow (\kappa, \omega)^2$.

⁵indeed, symmetric

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